Hamiltonian stationary Lagrangian tori contained in a hypersphere

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September 11, 2009

Abstract

The Clifford torus is a torus in a three-dimensional sphere. Homogeneous tori are simple generalization of the Clifford torus which still in a three-dimensional sphere. There is a way to construct tori in a three-dimensional sphere using the Hopf fibration. In this paper, all Hamiltonian stationary Lagrangian tori which is contained in a hypersphere in the complex Euclidean plane are constructed explicitly. Then it is shown that they are homogeneous tori. For the construction, flat quaternionic connections of Hamiltonian stationary Lagrangian tori are considered and a spectral curve of an associated family of them is used.

1 Introduction

We call a weakly-conformal immersion $f\colon M \to \mathbb{R}^4$ from a Riemann surface M to the four-dimensional Euclidean space $\mathbb{R}^4$ a *surface in* $\mathbb{R}^4$. The four-dimensional Euclidean space $\mathbb{R}^4$ is identified with the complex Euclidean plane $\mathbb{C}^2$. A surface in $\mathbb{C}^2$ is called *Hamiltonian stationary Lagrangian* if it is Lagrangian and stationary with respect to the area functional for every compactly supported Hamiltonian deformation family (see [10]).

There does not exist a Hamiltonian stationary Lagrangian sphere since the Lagrangian angle map is harmonic. A homogeneous torus $(x, y) \to (r_1 e^{xi}, r_2 e^{yi})$ $(r_1, r_2 > 0)$ is a simple example of a Hamiltonian stationary Lagrangian *unbranched* torus. A homogeneous torus is contained in a hypersphere. On the other hand, a homogeneous torus is a Hopf torus. A Hopf torus is an immersed torus in a hypersphere which is a lift of a closed and immersed curve in a two-dimensional sphere by the Hopf map [12]. If $r_1 = r_2$, then it is called the *Clifford torus*. Anciaux [1] showed that the Clifford torus is a Hamiltonian stationary Lagrangian torus which attains the minimum Willmore energy. The Willmore energy of a surface is the integral of the square norm of the mean curvature

*Partly supported by a Grant-in-Aid for Young Scientists (B) no. 19740028, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

vector. It is not known whether there exist Hamiltonian stationary Lagrangian torus whose image is contained in a hypersphere except homogeneous tori.

We will classify Hamiltonian stationary Lagrangian tori whose image is contained in a hypersphere.

Theorem 1.1. *A Hamiltonian stationary Lagrangian torus whose image is contained in a hypersphere is* (*a reparametrization of*) *a homogeneous torus or its covering.*

Hélein and Romon [7] classified all Hamiltonian stationary Lagrangian tori. Leschke and Romon [8] obtained a different classification. Hence it is theoretically possible to investigate everything about Hamiltonian stationary Lagrangian tori by these formula. But it will be difficult to accomplish it in practice according to circumstances.

Without the above formulas, we will prove Theorem 1.1. To prove our theorem, we use a flat quaternionic connection of a surface. The complex Euclidean plane $\mathbb{C}^2$ is identified with quaternions $\mathbb{H}$. When we consider $\mathbb{H}$ as a quaternionic vector space, a surface $f\colon M \to \mathbb{H}$ is a section of the trivial quaternionic line bundle H over M. If f does not vanish on M, then there exists a flat quaternionic connection ∇ on H such that $\nabla f = 0$. We call ∇ the *connection* of f.

Let $S^2(1)$ be a two-dimensional sphere of radius one. The Gauss map of f is a map from M to the direct product $S^2(1) \times S^2(1)$. We fix one of maps from M to $S^2(1) \subset \operatorname{Im}\mathbb{H}$ in the Gauss map. Then the map defines a smooth section J of a quaternionic endomorphism bundle $\operatorname{End}(H)$ of H. We see that $-J^2$ is the identity section. The section J is called a *complex structure* of f. The pair (H, J) is called a *complex* quaternionic line bundle of f. Then theory of surfaces becomes theory of parallel sections of flat quaternionic connections of (H, J). This is an interesting view point introduced in [5].

A *Willmore connection* of (H, J) defined in [5] is a flat quaternionic connection which is a critical point of the Willmore functional in the space of flat quaternionic connections. Every Willmore connection belongs to a family of Willmore connections parametrized by a circle. This family is called an *associated family*.

We show that if a flat quaternionic connection of (H, J) is a connection of a Hamiltonian stationary Lagrangian torus whose image is contained in a hypersphere, then it belongs to the associated family of the trivial quaternionic connection. Every parallel section of a connection in the associated family is a Hamiltonian stationary Lagrangian torus whose image is contained in a hypersphere. We construct a parallel section of a connection in the associated family of the trivial connection. Then a parallel section ψ is

$$\psi = \left(C_0 - \frac{m\delta_1 - (n\delta_0 - s)}{r\delta_1 - (r\delta_0 - s)} C_1 ij \right) \times \frac{1}{r\delta_1 - (r\delta_0 - s)i}$$

$$\times \left[\{(r-m)\delta_1 - [(r-m)\delta_0 - (s-n)]i\} \, e^{\pi\{(m+r)\delta_1 x - [(m+r)\delta_0 - (n+s)]y\}i/\delta_1} \right.$$
$$\left. + \{(r+m)\delta_1 - [(r+m)\delta_0 - (s+n)]i\} \, ie^{\pi\{(m-r)\delta_1 x - [(m-r)\delta_0 - (n-s)]y\}i/\delta_1} j \right],$$

where C_0, $C_1 \in \mathbb{C}$, δ_0, $\delta_1 \in \mathbb{R}$ such that $\mathbb{Z} + (\delta_0 + \delta_1 i)\mathbb{Z}$ is the lattice of a torus, and m, n, r, $s \in \mathbb{Z}$ such that

$$\left(m^2 - r^2\right) \delta_0^2 - 2(mn - rs)\delta_0 + (m^2 - r^2)\delta_1^2 + n^2 - s^2 = 0.$$

Then we have the classification.

2 Quaternionic formalism

We adopt the terminology of quaternionic formalism for surfaces in [11], [5], and [3] to fit our case.

Let $\mathbb{H}$ be the set of quaternions, that is the unitary real algebra generated by the symbols i, j, and k with relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ $(a_0, a_1, a_2, a_3 \in \mathbb{R})$, we denote by $\hat{a} = a_0 - a_1 i - a_2 j - a_3 k$ its quaternionic conjugate, $\operatorname{Re} a = a_0$ its real part, and $\operatorname{Im} a = a_1 i + a_2 j + a_3 k$ its imaginary part. We define a quaternionic hermitian product $\langle\ ,\ \rangle$ by $\langle a, b\rangle = a\hat{b}$.

For a_0 and $a_1 \in \mathbb{R}$, we consider $a = a_0 + a_1 i$ as a complex number. We should not that $\operatorname{Im} a = a_1 i$ by our definition. We denote by $\bar{a}$ its complex conjugate. We identify $\mathbb{H}$ with $\mathbb{R}^4$ and $\mathbb{C}^2$ by the following identification:

$$a_0, a_1, a_2, a_3 \in \mathbb{R},$$
$$a_0 + a_1 i + a_2 j + a_3 k \cong (a_0, a_1, a_2, a_3) \in \mathbb{R}^4,$$
$$a_0 + a_1 i + a_2 j + a_3 k \cong (a_0 + a_1 i, a_2 - a_3 i) \in \mathbb{C}^2,$$

Then $g(a, b) = \operatorname{Re}\langle a, b\rangle$ is the Euclidean inner product and $\Theta(a, b) = g(ai, b)$ is the symplectic form of $\mathbb{C}^2$. We use the same notation g and Θ for the Riemannian metric of $\mathbb{R}^4$ and the symplectic structure of $\mathbb{C}^2$ respectively.

We introduce *complex* quaternionic vector bundles to explain surfaces in terms of vector bundles. Let M be a Riemann surface with its complex structure J^M and V be a *left* quaternionic vector bundle over M. We denote by $\Gamma(V)$ the set of smooth sections of V and by V_p the fiber of V at p. Let $\Omega^n(V)$ be the set of V-valued n-forms on M for every non-negative integer n. Then $\Omega^0(V) = \Gamma(V)$. Let S be a smooth section of the quaternionic endomorphism bundle $\operatorname{End}(V)$ of V such that $-S_p \circ S_p$ is the identity map Id_p on V_p for every $p \in M$. A pair (V, S) is called a *complex quaternionic vector bundle* and S is called a *complex structure* of V. We have a splitting $\operatorname{End}(V) = \operatorname{End}(V)_+ \oplus \operatorname{End}(V)_-$, where

$$\operatorname{End}(V)_+ = \{\xi \in \operatorname{End}(V) : S\xi = \xi S\}, \ \ \operatorname{End}(V)_- = \{\xi \in \operatorname{End}(V) : S\xi = -\xi S\}.$$

This splitting induces a decomposition of $\xi \in \mathrm{End}(V)$ into $\xi = \xi_+ + \xi_-$, where $\xi_+ = (\xi - S\xi S)/2 \in \mathrm{End}(V)_+$ and $\xi_- = (\xi + S\xi S)/2 \in \mathrm{End}(V)_-$.

Let $T^*M \otimes_{\mathbb{R}} V$ be the tensor bundle of the cotangent bundle T^*M of M and V over real numbers. We set $*\omega = \omega \circ J^{TM}$ for every $\omega \in \Omega^1(V)$. We extend S to $T^*M \otimes_{\mathbb{R}} V$ by $S(\omega\phi) = \omega S(\phi)$ for every $\omega \in T^*M$ and every $\phi \in V$. We have a quaternionic linear splitting $T^*M \otimes_{\mathbb{R}} V = KV \oplus \bar{K}V$, where

$$KV = \{\eta \in T^*M \otimes_{\mathbb{R}} V : *\eta = S\eta\}, \ \ \bar{K}V = \{\eta \in T^*M \otimes_{\mathbb{R}} V : *\eta = -S\eta\}.$$

This splitting induces the *type decomposition* of $\eta \in T^*M \otimes_{\mathbb{R}} V$ into $\eta = \eta' + \eta''$, where $\eta' = (\eta - S * \eta)/2 \in KV$ and $\eta'' = (\eta + S * \eta)/2 \in \bar{K}V$.

We mainly use the left trivial quaternionic line bundle H over M. We identify a smooth map $\phi\colon M \to \mathbb{H}$ with a smooth section $p \mapsto (p, \phi(p))$ of H. The bundle $\mathrm{End}(H)$ is identified with H by the identification of $\xi \in \mathrm{End}(H)$ with $P \in H$ such that $\xi\phi = \phi P$ for every $\phi \in H$ where ξ, P and ϕ are in the fiber over the same point. Hence a complex structure $J \in \Gamma(\mathrm{End}(H))$ is identified with a smooth map $R\colon M \to \mathrm{Im}\,\mathbb{H}$ with $R^2 = -1$. The sets $\mathrm{End}(H)_+$ and $\mathrm{End}(H)_-$ are identified respectively with

$$H_+ = \{P \in H : RP = PR\}, \ \ H_- = \{P \in H : RP = -PR\}$$

since

$$J(\xi\phi) = J(\phi P) = -\phi PR, \ \ \xi(J\phi) = -\xi(\phi R) = -\phi RP$$

for every $\phi \in \Gamma(H)$. For $\xi \in \mathrm{End}(H)$, the components ξ_+ and ξ_- are identified with $P_+ = (P - RPR)/2$ and $P_- = (P + RPR)/2$ respectively. Then $T^*M \otimes_{\mathbb{R}} H$ decomposes as

$$T^*M \otimes_{\mathbb{R}} H = KH_+ \oplus KH_- \oplus \bar{K}H_+ \oplus \bar{K}H_-.$$

According to this decomposition, a quaternionic connection $\nabla\colon \Gamma(H) \to \Omega^1(H)$ decomposes as

$$\begin{gathered}
\nabla = \partial^\nabla + A^\nabla + \bar{\partial}^\nabla + Q^\nabla, \\
\nabla'\phi = (\nabla\phi)', \ \ \nabla''\phi = (\nabla\phi)'', \\
\partial^\nabla\phi = (\nabla\phi)'_+, \ \ A^\nabla\phi = (\nabla\phi)'_-, \ \ \bar{\partial}^\nabla\phi = (\nabla\phi)''_+, \ \ Q^\nabla\phi = (\nabla\phi)''_-,
\end{gathered}$$

where ϕ is any smooth section of H. We see that A^∇ and Q^∇ are tensorial, that is $A^\nabla \in \Gamma(K\,\mathrm{End}(H)_-)$ and $Q^\nabla \in \Gamma(\bar{K}\,\mathrm{End}(H)_-)$. The section A^∇ is called the *Hopf field of* ∇' and Q^∇ the *Hopf field of* ∇''. We have a splitting $(H, J) = E \oplus jE$, where

$$E = \{\phi \in H : J\phi = i\phi\}, \quad jE = \{j\phi \in H : J(j\phi) = (-i)j\phi\}.$$

The operator ∂^∇ is a complex anti-holomorphic structure of $E \oplus jE$ and $\bar{\partial}^\nabla$ is a complex holomorphic structure of $E \oplus jE$.

We call a weakly-conformal immersion from a Riemann surface to $\mathbb{R}^4$ a *surface*. We explain surfaces in terms of quaternions. Let $\phi\colon M \to \mathbb{H}$ be a non-constant smooth map. If there exists a smooth quaternionic-valued map $R\colon M \to \operatorname{Im}\mathbb{H}$ with $R^2 = -1$ such that $*(d\phi) = -(d\phi)R$, then ϕ is a surface. The converse does not hold in general. However, if ψ is a conformal immersion, then there exists a smooth quaternionic-valued map $R\colon M \to \operatorname{Im}\mathbb{H}$ with $R^2 = -1$ such that $*(d\psi) = -(d\psi)R$. The map R is called the *right normal vector* of ψ.

We describe surfaces in terms of complex quaternionic vector bundles. We define a complex structure $J \in \Gamma(\operatorname{End}(H))$ by $J\phi = -\phi R$ for every $\phi \in \Gamma(H)$. Then $d''\colon \Gamma(H) \to \Gamma(\bar{K}H)$ satisfies the equations

$$d''\phi = \frac{1}{2}[d\phi + J*(d\phi)], \quad d''(\lambda\phi) = \lambda(d''\phi) + [(d\lambda)\phi]''$$

for every $\phi \in \Gamma(H)$ and every smooth quaternionic-valued function λ on M. A smooth section ϕ of H satisfies the equation $d''\phi = 0$ if and only if ϕ is a constant map or a surface with its right normal vector R. We call d'' the *quaternionic holomorphic structure* of a surface with its right normal vector R. When we fix the map R firstly and do not assume the existence of ϕ with $d''\phi = 0$, we call d'' the quaternionic holomorphic structure of (H, J).

We relate a quaternionic holomorphic structure of a surface with a connection of (H, J). Let $\psi\colon M \to \mathbb{H}$ be a nowhere-vanishing surface with its right normal vector R. Then there exists a quaternionic connection on H with $\nabla\psi = 0$. We call ∇ the *connection* of ψ. Let $d\colon \Gamma(H) \to \Omega^1(H)$ be the trivial quaternionic connection on H. Then there exists $W \in \Omega^1(\operatorname{End}(H))$ such that $\nabla = d + W$. Since $\nabla''\psi = d''\psi + W''\psi$, $\nabla\psi = 0$, and $d''\psi = 0$, we have $W''\psi = 0$. Since ψ is nowhere-vanishing, we have $W'' = 0$, that is $W \in \Gamma(K\operatorname{End}(H))$ and $\nabla'' = d''$.

We define a quaternionic-valued one-form $\omega \in \Omega^1(H)$ by $\phi\omega = W\phi$ for every $\phi \in \Gamma(H)$. Then $\omega = -\psi^{-1}(d\psi) \in \Gamma(KH)$. The one-form ω is called the *connection form of* ∇ or the *connection form of* ψ. Since $d\omega - \omega \wedge \omega = 0$, the connection ∇ is flat.

The following is an example such that the connection form of a surface describe a geometric property of the surface.

Lemma 2.1. *Let ω be a connection form of a surface in $\mathbb{H}$. The image of the surface is contained in a hypersphere centered at the origin if and only if* $\operatorname{Re}\omega = 0$.

Proof. Let $\psi = \psi_0 + \psi_1 j\colon M \to \mathbb{H}$ be a surface with complex-valued functions ψ_0 and ψ_1 on M. Then

$$\omega = -\frac{1}{|\psi_0|^2 + |\psi_1|^2}\left\{\bar{\psi}_0(d\psi_0) + \psi_1(d\bar{\psi}_1) + \left[\bar{\psi}_0(d\psi_1) - \psi_1(d\bar{\psi}_0)\right]j\right\}.$$

We have

$$\begin{aligned}
\operatorname{Re}\omega =& -\frac{1}{|\psi_0|^2+|\psi_1|^2}\left[\bar{\psi}_0(d\psi_0)+\psi_1(d\bar{\psi}_1)+\psi_0(d\bar{\psi}_0)+\bar{\psi}_1(d\psi_1)\right] \\
=& -d\log\left(|\psi_0|^2+|\psi_1|^2\right).
\end{aligned}$$

Hence the $\psi(M)$ is contained in the hypersphere centered at the origin if and only if $\operatorname{Re}\omega = 0$. $\square$

We recall a *Willmore connection* which is used to construct a Hamiltonian stationary Lagrangian torus. Let ∇ be a flat quaternionic connection of a complex quaternionic line bundle (H, J) with $\nabla'' = d''$. We define a quaternionic-valued one-form $\alpha^\nabla \in \Gamma(KH_-)$ by $\phi\alpha^\nabla = A^\nabla\phi$ for every $\phi \in \Gamma(H)$. We call α^∇ the *one-form of a Hopf field* A^∇. Let $d^\nabla\colon \Omega^1(H) \to \Omega^2(H)$ be the covariant exterior derivative of ∇. We extend ∇ to the connection $\nabla\colon \operatorname{End}(H) \to \Omega^1(\operatorname{End}(H))$ of $\operatorname{End}(H)$ and d^∇ to the covariant exterior derivative $d^\nabla\colon \Omega^1(\operatorname{End}(H)) \to \Omega^2(\operatorname{End}(H))$ of $\operatorname{End}(H)$. Then ∇ is a *Willmore connection* if $d^\nabla * A^\nabla = 0$ (see [5, Lemma 6.1]).

We consider the condition such that the trivial quaternionic connection becomes a Willmore connection. The trivial quaternionic connection d is flat. The Hopf field of d' satisfies the equation

$$\begin{aligned}
A^d\phi =& \frac{1}{2}\left[(d' + Jd'J)\right]\phi \\
=& \frac{1}{4}\left[d - J*d + J(d - J*d)J\right]\phi \\
=& \frac{1}{4}\{(d\phi) + *(d\phi)R \\
& + [-(d\phi)R - \phi(dR)](-R) + [-*(d\phi)R - \phi*(dR)]\} \\
=& \frac{1}{4}\phi\left[(dR)R - *(dR)\right] = \phi\alpha^d
\end{aligned} \tag{1}$$

for every $\phi \in \Gamma(H)$. Hence d is a Willmore connection on (H, J) if and only if

$$[d*(dR)]R - *(dR)\wedge(dR) = 0.$$

We define an associated family of a connection which is a main tool for the construction of Hamiltonian stationary Lagrangian tori. Let ∇ be a Willmore connection of (H, J) with $\nabla'' = d''$. When $A^\nabla \neq 0$, we call the family of quaternionic connections

$$\left\{\nabla^\theta = \nabla - A^\nabla + (\cos\theta)A^\nabla + (\sin\theta)JA^\nabla : \theta \in \mathbb{R}\right\}$$

the *associated family* of ∇. We see that ∇^θ is Willmore for every θ by [5, Lemma 6.2], that $\left(\nabla^\theta\right)'' = d''$, and that $\partial^{\nabla^\theta} = \partial^\nabla$.

3 Lagrangian surfaces

We review the quaternionic formalism for Lagrangian surfaces.

A surface $\psi\colon M \to \mathbb{C}^2$ is called a *Lagrangian surface* if $\psi^*\Theta = 0$, where Θ is the symplectic structure of $\mathbb{C}^2$. Let $\mathbb{Z}$ be the set of integers. Then ψ is a Lagrangian surface if and only if the right normal vector of ψ is $e^{-\beta i}j$ with a smooth map $\beta\colon M \to \mathbb{R}/2\pi\mathbb{Z}$ by [6]. If β is constant, then ψ is a holomorphic map with respect to a complex structure of $\mathbb{R}^4$ and the image of ψ is a Lagrangian plane. Hence if M is closed, then β is non-constant.

We define a complex structure J of H by $J\phi = -\phi R = -\phi e^{-\beta i}j$ for every $\phi \in H$. Then the quaternionic holomorphic structure of (H, J) satisfies the equation

$$d''\phi = \frac{1}{2}\left[(d\phi) - *(d\phi)e^{-\beta i}j\right]$$

and the Hopf field satisfies the equation

$$A^d\phi = \phi\frac{1}{4}\left[(d\beta)i + *(d\beta)ie^{-\beta i}j\right] = \phi\alpha^d \tag{2}$$

for every $\phi \in \Gamma(H)$ by the equation (1).

We will recall the definition of *Hamiltonian stationary Lagrangian surfaces* introduced in [10]. Let $\psi\colon M \to \mathbb{H}$ be a Lagrangian surface with its right normal vector $e^{-\beta i}j$. A smooth family $\{\psi_t\colon M \to \mathbb{H}\}_{t\in I}$ of surfaces such that $\psi = \psi_0$ parametrized by an interval I containing zero is called a *deformation family* of ψ. Let $V_t = d\psi_t/dt \in \Gamma(\psi_t^* T\mathbb{H})$. A deformation family $\{\psi_t\}$ of ψ is called *Hamiltonian* if $\psi_t^*(V_t \lrcorner \Theta)$ is an exact one-form on M for every $t \in I$. A Lagrangian surface ψ is said to be *Hamiltonian stationary* if ψ is stationary with respect to the area functional for every compactly supported Hamiltonian deformation family. The map $\beta\colon M \to \mathbb{R}/2\pi\mathbb{Z}$ is a harmonic map if and only if ψ is Hamiltonian stationary by Theorem 1 in [7]. Let $R = e^{-\beta i}j$. Then

$$[d * (dR)]R - *(dR) \wedge (dR) = (d * d\beta)i.$$

Hence the trivial quaternionic connection d is a Willmore connection of (H, J) for every Hamiltonian stationary Lagrangian surface.

We will write a harmonic map β from a torus to $\mathbb{R}/2\pi\mathbb{Z}$ explicitly. For every $\delta = \delta_0 + \delta_1 i \in \mathbb{C} \setminus \{0\}$ with real numbers δ_0 and δ_1, we define a torus $\mathcal{T}_\delta$ by $\mathcal{T}_\delta = \mathbb{C}/\Lambda_\delta$ with $\Lambda_\delta = \{m + n\delta : m, n \in \mathbb{Z}\}$. We define a set $F \subset \mathbb{C}$ by

$$F = \{\delta_0 + \delta_1 i : \delta_0, \delta_1 \in \mathbb{R},\, \delta_0^2 + \delta_1^2 > 1,\, -1/2 < \delta_0 < 1/2,\, \delta_1 > 0\}.$$

Every torus is conformally equivalent to $\mathcal{T}_\delta$ for some δ in the closure of F. We denote by (x, y) the real coordinate of $\mathbb{C}$ such that $z = x + yi$ is the standard holomorphic coordinate of $\mathbb{C}$. We consider a map from $\mathbb{C}$ which is periodic with respect to Λ_δ as a map from $\mathcal{T}_\delta$. We assume that $\beta\colon \mathcal{T}_\delta \to \mathbb{R}/2\pi\mathbb{Z}$ is a harmonic map. Then $(d\beta)$ is a harmonic differential on $\mathcal{T}_\delta$ and $(d\beta) - i * (d\beta)$ is a complex

holomorphic one-form on $\mathcal{T}_\delta$ (see [4]). Hence there exist real numbers a_0 and a_1 such that

$$(d\beta) - i*(d\beta) = (a_0 + a_1 i)(dz) = a_0(dx) - a_1(dy) + [a_0(dy) + a_1(dx)]i.$$

Then $(d\beta) = a_0(dx) - a_1(dy)$. Changing the coordinate suitably, we may assume that $\beta = a_0 x - a_1 y$. Since β is a smooth map to $\mathbb{R}/2\pi\mathbb{Z}$, real numbers a_0 and a_1 satisfy the equation $a_0 m + (a_0\delta_0 - a_1\delta_1)n \in 2\pi\mathbb{Z}$ for every m, $n \in \mathbb{Z}$. Hence $a_0 = 2\pi r$, $a_1 = 2\pi(r\delta_0 - s)/\delta_1$, and

$$\beta = 2\pi r x - \frac{2\pi(r\delta_0 - s)}{\delta_1} y \tag{3}$$

with r and $s \in \mathbb{Z}$.

We give an example of a Hamiltonian stationary Lagrangian torus. A map $f\colon \mathcal{T}_{\delta_1 i} \to \mathbb{H}$ defined by

$$f = r\left(e^{2\pi x i} + j\delta_1 e^{2\pi y i/\delta_1}\right) \quad (r > 0) \tag{4}$$

is called a *homogeneous torus*. A homogeneous torus f is a Hamiltonian stationary Lagrangian torus with its right normal vector $e^{\beta i}j = e^{2\pi(x+y/\delta_1)i}j$. If $\delta_1 = 1$, then f is called the *Clifford torus*.

We have

$$\begin{aligned} f^{-1} &= \frac{1}{r^2(1+\delta_1^2)}\left(e^{-2\pi x i} - j\delta_1 e^{2\pi y i/\delta_1}\right), \\ df &= 2\pi r\left[e^{2\pi x i} i(dx) - i e^{-2\pi y i/\delta_1}(dy) j\right]. \end{aligned}$$

Hence the connection form ω of f is

$$\begin{aligned} \omega &= -f^{-1}(df) \\ &= -\frac{2\pi}{r(1+\delta_1^2)}\left\{i\left[(dx) + \delta_1(dy)\right] + i\left[-(dy) + \delta_1(dx)\right]e^{-2\pi(x+y/\delta_1)i}j\right\} \\ &= -\frac{2\pi}{r(1+\delta_1^2)}\left[\left(d\tilde{\beta}\right)i + *\left(d\tilde{\beta}\right)ie^{-\beta i}j\right] \end{aligned}$$

$$\tilde{\beta} = x + \delta_1 y.$$

4 Connections

We give a relation between Hamiltonian stationary Lagrangian surfaces and associated families.

Lemma 4.1. *A connection of a Hamiltonian stationary Lagrangian surface in the complex Euclidean plane whose image is contained in a hypersphere centered at the origin belongs to the associated family of the trivial quaternionic connection.*

Proof. Let $\psi\colon M \to \mathbb{H}$ be a Hamiltonian stationary Lagrangian surface from M with its right normal vector $e^{-\beta i}j$ and ∇ the connection of ψ. We decompose the connection form ω of ∇ as $\omega = \omega_0 + \omega_1 j$ with complex-valued one-forms ω_0 and ω_1 on M. Since $\nabla'' = d''$, we have

$$*\omega = *\omega_0 + *\omega_1 j = -\omega R = \omega_1 e^{\beta i} - \omega_0 e^{-\beta i} j.$$

Hence $\omega_1 = *\omega_0 e^{-\beta i}$ and $\omega = \omega_0 + *\omega_0 e^{-\beta i} j$.

We assume that $\psi(M)$ is contained in the hypersphere centered at the origin. Then $\mathrm{Re}\,\omega = 0$ by Lemma 2.1. Then $\omega = \xi i + *\xi i e^{-\beta i} j$ with a real-valued one-form ξ on M. If $\xi = 0$, then $\nabla = d$. We assume that $\xi \neq 0$. Then

$$d\omega = d\xi i + (d*\xi) i e^{-\beta i} j - *\xi \wedge (d\beta) e^{-\beta i} j,$$
$$\omega \wedge \omega = -2\xi \wedge *\xi e^{-\beta i} j.$$

Since $d\omega - \omega \wedge \omega = 0$, we have

$$d\xi = d*\xi = 0, \ \ [(d\beta) + 2\xi] \wedge *\xi = 0.$$

Hence $(d\beta) + 2\xi = \kappa *\xi$ with a real-valued function κ on M. Since β is harmonic, we have $(d\kappa) \wedge *\xi = (d\kappa) \wedge \xi = 0$. Hence κ is a constant. Since

$$2\xi - \kappa * \xi = -(d\beta), \quad \kappa\xi + 2*\xi = -*(d\beta),$$

we have

$$\xi = -\frac{1}{\kappa^2+4}[2(d\beta) + \kappa*(d\beta)],$$
$$\omega = -\frac{1}{\kappa^2+4}[2(d\beta) + \kappa*(d\beta)]i - \frac{1}{\kappa^2+4}[2*(d\beta) - \kappa(d\beta)] i e^{-\beta i} j.$$

Let $\kappa = 2\tan(\theta/2)$. Then

$$\begin{aligned}
\omega = &-\frac{1}{4}\left(\cos\frac{\theta}{2}\right)^2 \left[2(d\beta) + 2\left(\tan\frac{\theta}{2}\right)*(d\beta)\right] i \\
&-\frac{1}{4}\left(\cos\frac{\theta}{2}\right)^2 \left[2*(d\beta) - 2\left(\tan\frac{\theta}{2}\right)(d\beta)\right] i e^{-\beta i} j \\
= &-\frac{1}{4}\left[(\cos\theta + 1)(d\beta) + (\sin\theta)*(d\beta)\right] i \\
&-\frac{1}{4}\left[(\cos\theta + 1)*(d\beta) - (\sin\theta)(d\beta)\right] i e^{-\beta i} j \\
= &-\frac{1}{4}\left[(d\beta) i + *(d\beta) i e^{-\beta i} j\right] \\
&-\frac{\cos\theta}{4}\left[(d\beta) i + *(d\beta) i e^{-\beta i} j\right] + \frac{\sin\theta}{4}\left[(d\beta) i + *(d\beta) i e^{-\beta i} j\right] e^{-\beta i} j.
\end{aligned}$$

Hence $\omega = -\alpha^d - (\cos\theta)\alpha^d + (\sin\theta)\alpha^d e^{-\beta i} j$. The associated family of the trivial connection is

$$\left\{ d - A^d + (\cos\theta) A^d + (\sin\theta) J A^d : \theta \in \mathbb{R} \right\}.$$

Then ∇ belongs to the associated family of d. $\square$

5 The Dirac operator

We give another system of equations for a quaternionic holomorphic section of (H,J) with respect to d''. We use this system of equations later.

In the beginning, we will obtain a frame of E. We assume that M is a closed Riemann surface which is not necessarily a torus and that $\beta\colon M\to\mathbb{R}/2\pi\mathbb{Z}$ is a non-constant smooth map. We define a complex quaternionic line bundle (H,J) over M by $J\phi=-\phi e^{-\beta i}j$ for every $\phi\in\Gamma(H)$. Let ϕ_0 and ϕ_1 be complex-valued functions on M such that the map $\phi=\phi_0+\phi_1 j$ is a smooth section of a complex line bundle E with $H=E\oplus jE$. Since $-(\phi_0+\phi_1 j)e^{-\beta i}j=i(\phi_0+\phi_1 j)$, we have $\phi_1=\phi_0 ie^{-\beta i}$. Hence $\phi=\phi_0\left(1+ie^{-\beta i}j\right)$.

We will determine a complex holomorphic section of E with respect to $\bar\partial^d$. Let λ be a smooth complex-valued function on M and $\mu=1+ie^{-\beta i}j$. Then $\lambda\mu$ is a smooth section of E and

$$\begin{aligned}
d''\mu&=\frac{1}{2}\left[*(d\beta)+(d\beta)e^{-\beta i}j\right],\\
d''(\lambda\mu)&=\frac{\lambda}{2}\left[*(d\beta)+(d\beta)e^{-\beta i}j\right]+\left(\bar\partial\lambda\right)\mu,\\
Jd''J(\lambda\mu)=&Jd''(i\lambda\mu)=J\{i\lambda d''(\mu)+\left[\bar\partial(i\lambda)\right]\mu\}\\
=&-\frac{i\lambda}{2}\left[*(d\beta)+(d\beta)e^{-\beta i}j\right]e^{-\beta i}j-\left(\bar\partial\lambda\right)\mu\\
=&\frac{\lambda}{2}\left[(d\beta)i-*(d\beta)ie^{-\beta i}j\right]-\left(\bar\partial\lambda\right)\mu.
\end{aligned}$$

Hence

$$\begin{aligned}
\bar\partial^d(\lambda\mu)&=\frac{\lambda}{2}\left(\bar\partial\beta\right)\left(-i+e^{-\beta i}j\right)+\left(\bar\partial\lambda\right)\mu=\left[-\frac{\lambda i}{2}\left(\bar\partial\beta\right)+\left(\bar\partial\lambda\right)\right]\mu,\\
Q^d(\lambda\mu)&=\frac{\lambda}{2}(\partial\beta)\left(i+e^{-\beta i}j\right)=\frac{\lambda}{2}(\partial\beta)e^{-\beta i}j\mu.
\end{aligned}$$

We see that $\bar\partial^d(\lambda\mu)=0$ if and only if $\lambda=Ce^{\beta i/2}$ with $C\in\mathbb{C}$. Hence $\theta=e^{\beta i/2}\mu=e^{\beta i/2}+ie^{-\beta i/2}j$ satisfies the equation $d''\theta=0$.

We see that $(\theta,j\theta)$ is a complex holomorphic frame of (E,jE) with respect to $\bar\partial^d$. We will write the equation $d''\psi=0$ in terms of this frame. We have

$$\bar\partial^d(\lambda\theta)=\left(\bar\partial\lambda\right)\theta,\quad Q^d(\lambda\theta)=\frac{\lambda}{2}(\partial\beta)j\theta$$

for arbitrary complex-valued function λ on M. Similarly, we have

$$\bar\partial^d(\lambda j\theta)=(\partial\lambda)j\theta,\quad Q^d(\lambda j\theta)=-\frac{\lambda}{2}\left(\bar\partial\beta\right)\theta.$$

Hence the equation $d''(\lambda_0\theta+\lambda_1 j\theta)=0$ with complex-valued functions λ_0 and λ_1 on M becomes the equation

$$\left[\bar\partial\lambda_0-\frac{\lambda_1}{2}\left(\bar\partial\beta\right)\right]\theta+\left[\partial\lambda_1+\frac{\lambda_0}{2}\left(\partial\beta\right)\right]j\theta=0,$$

that is

$$\left[\begin{pmatrix}\bar\partial & 0\\ 0 & \partial\end{pmatrix}+\begin{pmatrix}0 & -\left(\bar\partial\beta\right)/2\\ (\partial\beta)/2 & 0\end{pmatrix}\right]\begin{pmatrix}\lambda_0\\ \lambda_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

This equation is equivalent to the equation

$$\left[\begin{pmatrix}0 & \partial\\ -\bar\partial & 0\end{pmatrix}+\begin{pmatrix}(\partial\beta)/2 & 0\\ 0 & \left(\bar\partial\beta\right)/2\end{pmatrix}\right]\begin{pmatrix}\lambda_0\\ \lambda_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

We assume that $M=\mathcal{T}_\delta$ and that β is defined by (3). We define a Dirac operator $\mathcal{D}$ with potential on $\mathbb{C}$ by

$$\mathcal{D}=\begin{pmatrix}0 & \partial/\partial z\\ -\partial/\partial\bar z & 0\end{pmatrix}+\begin{pmatrix}\beta_z/2 & 0\\ 0 & \beta_{\bar z}/2\end{pmatrix}.$$

The solution to the equation

$$\mathcal{D}\begin{pmatrix}\lambda_0\\ \lambda_1\end{pmatrix}=0$$

on $\mathbb{C}$ which is periodic with respect to Λ_δ is considered as a solution to the equation on $\mathcal{T}_\delta$.

6 Spectral curves

We will show that a branched double covering of $\mathbb{C}P^1$ with two branched points is accompanied with the associated family of the trivial connection.

We define $\mathcal{I}$ by $\mathcal{I}\psi=i\psi$ for every $\psi\in\mathbb{H}$. By the complex structure $\mathcal{I}$, we consider H as a trivial complex vector bundle C^2 of rank two. Then $H=C\oplus Cj$, where C is a complex trivial line bundle. We consider a complex hermitian product $(\ ,\ )$ of C^2 defined by $(\lambda_0+\lambda_1 j,\mu_0+\mu_1 j)=\lambda_0\bar\mu_0+\lambda_1\bar\mu_1$ for every λ_0, λ_1, μ_0, $\mu_1\in C$. A pair $(\epsilon_0,\epsilon_1)=(1,j)$ is a unitary frame of C^2. We have

$$\mathcal{I}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix}=\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix},\quad J\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix}=\begin{pmatrix}0 & -e^{-\beta i}\\ e^{\beta i} & 0\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix},$$
$$A^d\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}(d\beta)i & *(d\beta)ie^{-\beta i}\\ *(d\beta)ie^{\beta i} & -(d\beta)i\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix}.$$

Hence the associated family $\{d^\theta\}$ is a family of complex special linear connections. Let

$$\cos\theta=\frac{1}{2}\left(\frac{1}{\zeta}+\zeta\right),\quad \sin\theta=\frac{1}{2}\left(\frac{1}{\zeta}-\zeta\right)i\quad(\zeta\in\mathbb{C},\ |\zeta|=1).$$

Then

$$d^\theta=d-A^d+\frac{1}{2\zeta}(1+\mathcal{I}J)A^d+\frac{\zeta}{2}(1-\mathcal{I}J)A^d=\nabla_\zeta.$$

We extend the parameter domain $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$ of the associated family to $\mathbb{C} \setminus \{0\}$. We call the family of connection $\{\nabla_\zeta : \zeta \in \mathbb{C} \setminus \{0\}\}$ the *extended family* of $\{d^\theta\}$. We have

$$
\begin{aligned}
\mathcal{I}JA^d \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix} &= \frac{1}{4} \begin{pmatrix} *(d\beta) & -(d\beta)e^{-\beta i} \\ -(d\beta)e^{\beta i} & -*(d\beta) \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix}, \\
\frac{1}{2}(1+\mathcal{I}J)A^d &= \frac{\partial\beta}{4} \begin{pmatrix} i & -e^{-\beta i} \\ -e^{\beta i} & -i \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix}, \\
\frac{1}{2}(1-\mathcal{I}J)A^d &= \frac{\bar\partial\beta}{4} \begin{pmatrix} i & e^{-\beta i} \\ e^{\beta i} & -i \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix}, \\
\partial &= \frac{1}{2}(d - i*d), \quad \bar\partial = \frac{1}{2}(d + i*d).
\end{aligned}
$$

Let $\Phi = (1+\mathcal{I}J)A^d/2$. Then the conjugate transpose Φ^* of Φ is $-(1-\mathcal{I}J)A^d/2$ and $A^d = \Phi - \Phi^*$. Hence

$$
\nabla_\zeta = d - A^d + \frac{1}{\zeta}\Phi - \zeta\Phi^* = d + \left(\frac{1}{\zeta} - 1\right)\Phi - (\zeta - 1)\Phi^* \quad (\zeta \in \mathbb{C} \setminus \{0\}).
$$

A connection ∇_ζ is flat for every $\zeta \in \mathbb{C} \setminus \{0\}$ by Lemma 6.3 in [5] and $\nabla_1 = d$. We have

$$
\begin{aligned}
\nabla_\zeta \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix} = \Bigg[&\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} + \frac{\partial\beta}{4}\left(\frac{1}{\zeta} - 1\right) \begin{pmatrix} i & -e^{-\beta i} \\ -e^{\beta i} & -i \end{pmatrix} \\
&- \frac{\bar\partial\beta}{4}(\zeta - 1) \begin{pmatrix} -i & -e^{-\beta i} \\ -e^{\beta i} & i \end{pmatrix} \Bigg] \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix}.
\end{aligned}
$$

We take a covering of $\mathcal{T}_\delta$. Let $\tilde\Lambda_\delta = \{2m + 2n\delta : m, n \in \mathbb{Z}\}$ and $\tilde{\mathcal{T}}_\delta = \mathbb{C}/\tilde\Lambda_\delta$. Let $\tilde\rho\colon \tilde{\mathcal{T}}_\delta \to \mathcal{T}_\delta$ be the projection and $\tilde\rho^* H$ the pull-back of H by $\tilde\rho$. The torus $\mathcal{T}_\delta$ is biholomorphic to the torus $\tilde{\mathcal{T}}_\delta$.

We fix a special frame of $\tilde\rho^* H$ to make the discussion simple. Let

$$
h(z) \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix} = \begin{pmatrix} e^{\beta i/2} & 0 \\ 0 & e^{-\beta i/2} \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix}.
$$

Then h is a gauge transformation of $\tilde\rho^* H$ since

$$
h(z + m + \delta n) = (-1)^{mr+ns} h(z)
$$

for every n, $m \in \mathbb{Z}$. Let

$$
\epsilon = \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix}, \quad \tilde\epsilon = \begin{pmatrix} \tilde\epsilon_0 \\ \tilde\epsilon_1 \end{pmatrix} = h \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \end{pmatrix} = h\epsilon,
$$

and B the matrix-valued connection form of $\tilde\nabla_\zeta = \tilde\rho^* \nabla_\zeta$ with respect to $\tilde\epsilon$. Then

$$
\tilde\nabla_\zeta \tilde\epsilon = \tilde\nabla_\zeta h\epsilon = \left\{ dh + h\left[\left(\frac{1}{\zeta} - 1\right)\Phi - (\zeta - 1)\Phi^*\right]\right\}\epsilon
$$

$$= \left\{ dh + h \left[\left(\frac{1}{\zeta} - 1 \right) \Phi - (\zeta - 1)\Phi^* \right] \right\} h^{-1}\tilde{\epsilon} = B\tilde{\epsilon}.$$

Hence

$$\begin{aligned}
B =&(dh)h^{-1} + h \left[\left(\frac{1}{\zeta} - 1 \right) \Phi - (\zeta - 1)\Phi^* \right] h^{-1} \\
=&\frac{d\beta}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \frac{\partial\beta}{4} \left(\frac{1}{\zeta} - 1 \right) \begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix} - \frac{\bar{\partial}\beta}{4}(\zeta - 1) \begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix} \\
=&\frac{\partial\beta}{4} \begin{pmatrix} i(1+\zeta^{-1}) & 1-\zeta^{-1} \\ 1-\zeta^{-1} & -i(1+\zeta^{-1}) \end{pmatrix} + \frac{\bar{\partial}\beta}{4} \begin{pmatrix} i(\zeta+1) & \zeta-1 \\ \zeta-1 & -i(\zeta+1) \end{pmatrix}.
\end{aligned}$$

In the following, we will fix the frame $\tilde{\epsilon}$. Considering $\tilde{\nabla}_\zeta$ as a connection on H, we calculate the holonomy matrices of $\tilde{\nabla}_\zeta$. Let $\gamma_0(t) = t$ and $\gamma_1(t) = \delta t$. Then $\gamma_0([0,1])$ and $\gamma_1([0,1])$ are generators of the fundamental group $\pi_1(\mathcal{T}_\delta)$ of $\mathcal{T}_\delta$. Then

$$\begin{aligned}
B\left(\dot{\gamma}_0(t)\right) =&\frac{\beta_z}{4} \begin{pmatrix} i(1+\zeta^{-1}) & 1-\zeta^{-1} \\ 1-\zeta^{-1} & -i(1+\zeta^{-1}) \end{pmatrix} + \frac{\beta_{\bar{z}}}{4} \begin{pmatrix} i(\zeta+1) & \zeta-1 \\ \zeta-1 & -i(\zeta+1) \end{pmatrix} \\
=&\frac{\beta_{\bar{z}}\zeta + \beta_z}{4\zeta} \begin{pmatrix} i(\zeta+1) & \zeta-1 \\ \zeta-1 & -i(\zeta+1) \end{pmatrix}, \\
B\left(\dot{\gamma}_1(t)\right) =&\frac{\beta_z\delta}{4} \begin{pmatrix} i(1+\zeta^{-1}) & 1-\zeta^{-1} \\ 1-\zeta^{-1} & -i(1+\zeta^{-1}) \end{pmatrix} + \frac{\beta_{\bar{z}}\bar{\delta}}{4} \begin{pmatrix} i(\zeta+1) & \zeta-1 \\ \zeta-1 & -i(\zeta+1) \end{pmatrix} \\
=&\frac{\beta_{\bar{z}}\delta\zeta + \beta_z\bar{\delta}}{4\zeta} \begin{pmatrix} i(\zeta+1) & \zeta-1 \\ \zeta-1 & -i(\zeta+1) \end{pmatrix},
\end{aligned}$$

$$\beta_x = \frac{\partial\beta}{\partial x}, \quad \beta_y = \frac{\partial\beta}{\partial y},$$

$$\beta_z = \frac{1}{2}\left(\beta_x - \beta_y i\right), \quad \beta_{\bar{z}} = \frac{1}{2}\left(\beta_x + \beta_y i\right).$$

The matrices $G_0(\zeta)$ and $G_1(\zeta)$ defined by

$$G_m(\zeta) = \exp[-B(\dot{\gamma}_m(1))] \quad (m = 0, 1)$$

are the holonomy matrices with the base point $\rho(0)$.

We will calculate the trace of them by the following lemma.

Lemma 6.1. *Let*

$$G = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \left(a^2 + b^2 \neq 0, \ \ a, b \in \mathbb{C}\right).$$

Then

$$\begin{aligned}
\exp G =& \sum_{n=0}^{\infty} \frac{(a^2+b^2)^n}{(2n)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{(a^2+b^2)^n}{(2n+1)!} G \\
=& \cosh\sqrt{a^2+b^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} G.
\end{aligned}$$

The eigenvalues of $\exp G$ *is* $e^{\sqrt{a^2+b^2}}$ *and* $e^{-\sqrt{a^2+b^2}}$.

A simple calculation proves this lemma.

The trace of the holonomy matrices $G_0(\zeta)$ and $G_1(\zeta)$ are respectively

$$\begin{aligned} g_0(\zeta) =& 2\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left[\frac{-(\beta_{\bar z}\zeta+\beta_z)^2}{4\zeta}\right]^n = 2\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left[\frac{(\beta_{\bar z}\zeta+\beta_z)i}{2\sqrt{\zeta}}\right]^{2n} \\ =& 2\cosh\frac{(\beta_{\bar z}\zeta+\beta_z)i}{2\sqrt{\zeta}} = 2\cos\frac{\beta_{\bar z}\zeta+\beta_z}{2\sqrt{\zeta}}, \\ g_1(\zeta) =& 2\cos\frac{\beta_{\bar z}\bar\delta\zeta+\beta_z\delta}{2\sqrt{\zeta}}. \end{aligned}$$

Since

$$[g_0(\zeta)]^2-4=-4\left(\sin\frac{\beta_{\bar z}\zeta+\beta_z}{2\sqrt{\zeta}}\right)^2,$$

$$\frac{d}{d\zeta}\left(\sin\frac{\beta_{\bar z}\zeta+\beta_z}{2\sqrt{\zeta}}\right)^2 = -4\frac{\beta_{\bar z}\zeta-\beta_z}{\zeta\sqrt{\zeta}}\sin\frac{\beta_{\bar z}\zeta+\beta_z}{2\sqrt{\zeta}}\cos\frac{\beta_{\bar z}\zeta+\beta_z}{2\sqrt{\zeta}},$$

$$\begin{aligned} &\frac{d^2}{d\zeta^2}\left(\sin\frac{\beta_{\bar z}\zeta+\beta_z}{2\sqrt{\zeta}}\right)^2 \\ &= -\frac{1}{2\zeta^3}\left[(\beta_{\bar z}\zeta-\beta_z)^2\cos\frac{\beta_{\bar z}\zeta+\beta_z}{\sqrt{\zeta}} - \sqrt{\zeta}(\beta_{\bar z}\zeta-3\beta_z)\sin\frac{\beta_{\bar z}\zeta+\beta_z}{\sqrt{\zeta}}\right], \end{aligned}$$

the order of every zero of $[h_0(\zeta)]^2-4$ is two. We define a complex curve Σ by

$$\Sigma = \left\{(\zeta,\eta)\in(\mathbb{C}\setminus\{0\})\times(\mathbb{C}\setminus\{0\}) : \eta^2=\zeta\right\}.$$

The curve Σ is a double covering of $\mathbb{C}\setminus\{0\}$. The traces of the holonomy matrices are single-valued on Σ. We consider objects defined at $\zeta\in\mathbb{C}\setminus\{0\}$ as objects defined at $(\eta^2,\eta)\in\Sigma$. Then the associated family is $\left\{\nabla_{\eta^2} : |\eta|=1\right\}$.

We define a complex curve $\bar\Sigma$ by

$$\bar\Sigma = \left\{(\zeta,\eta)\in(\mathbb{C}\cup\{\infty\})\times(\mathbb{C}\cup\{\infty\}) : \eta^2=\zeta\right\}.$$

The complex curve $\bar\Sigma$ is called the *spectral curve* of the associated family $\left\{\nabla_{\eta^2}\right\}$ in [5].

7 Asymptotic behavior

A parallel section of a connection in the associated family $\{\nabla_{\eta^2} : |\eta|=1\}$ is a Hamiltonian stationary Lagrangian torus. We show that a parallel section of ∇_{η^2} has special asymptotic behavior at $\eta=0$ and at $\eta=\infty$.

We consider $\tilde\nabla_{\eta^2}$ as a connection on $\mathbb{C}$. Let $u\colon \mathbb{C}\times\bar\Sigma\to\mathbb{C}$ be the projection. Then a parallel section of $u^*\tilde\nabla_{\eta^2}$ is considered as a parallel section of $\tilde\nabla_{\eta^2}$.

Lemma 7.1. *Let $\psi_0 = \psi_0(z,\eta)$ and $\psi_1 = \psi_1(z,\eta)$ be complex-valued functions on $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$ such that $\psi_0\epsilon_0 + \psi_1\epsilon_1$ is a parallel section of $u^*\tilde{\nabla}_{\eta^2}$. Then*

$$\begin{aligned}
&\begin{pmatrix}\psi_0 & \psi_1\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix} = \begin{pmatrix}\psi_0 & \psi_1\end{pmatrix}\begin{pmatrix}e^{-\beta i/2} & 0\\ 0 & e^{\beta i/2}\end{pmatrix}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix}\\
&= \sum_{m,n=0}^{1}\begin{pmatrix}p_{m,n,0} & p_{m,n,1}\end{pmatrix} e^{[(-1)^m\beta_z i/(2\eta)]z + [(-1)^n\beta_{\bar{z}} i\eta/2]\bar{z}}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix},
\end{aligned}$$

where $p_{m,n,l} = p_{m,n,l}(\eta)$ is a complex-valued function on $\mathbb{C}\setminus\{0\}$ $(l,m,n = 0,1)$.

Proof. We decompose $\tilde{\nabla}_{\eta^2}$ as follows:

$$\begin{gathered}
\tilde{\nabla}_{\eta^2} = \tilde{\nabla}^{(1,0)}_{\eta^2} + \tilde{\nabla}^{(0,1)}_{\eta^2},\\
\tilde{\nabla}^{(1,0)}_{\eta^2}\tilde{\epsilon} = \left[\begin{pmatrix}\partial & 0\\ 0 & \partial\end{pmatrix} + \frac{\partial\beta}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} + \frac{\partial\beta}{4}\left(\frac{1}{\eta^2} - 1\right)\begin{pmatrix}i & -1\\ -1 & -i\end{pmatrix}\right]\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix},\\
\tilde{\nabla}^{(0,1)}_{\eta^2}\tilde{\epsilon} = \left[\begin{pmatrix}\bar{\partial} & 0\\ 0 & \bar{\partial}\end{pmatrix} + \frac{\bar{\partial}\beta}{2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \frac{\bar{\partial}\beta}{4}\left(\eta^2 - 1\right)\begin{pmatrix}-i & -1\\ -1 & i\end{pmatrix}\right]\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix}.
\end{gathered}$$

Let

$$\check{\epsilon} = \begin{pmatrix}\check{\epsilon}_0\\ \check{\epsilon}_1\end{pmatrix} = \begin{pmatrix}\beta_z i/4 & -\beta_z/4\\ 1 & 0\end{pmatrix}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix}.$$

Then

$$\tilde{\nabla}^{(1,0)}_{\eta^2}\check{\epsilon} = \left\{\begin{pmatrix}\partial & 0\\ 0 & \partial\end{pmatrix} + \left[\frac{1}{\eta^2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \begin{pmatrix}-\beta_z i/2 & -\beta_z^2/4\\ -1 & \beta_z i/2\end{pmatrix}\right](dz)\right\}\begin{pmatrix}\check{\epsilon}_0\\ \check{\epsilon}_1\end{pmatrix}.$$

We assume that $\check{\sigma}_0\check{\epsilon}_0 + \check{\sigma}_1\check{\epsilon}_1 = \check{\sigma}_0(z,\eta)\check{\epsilon}_0 + \check{\sigma}_1(z,\eta)\check{\epsilon}_1$ is a solution to the differential equation $u^*\tilde{\nabla}^{(1,0)}_{\eta^2}(\check{\sigma}_0\check{\epsilon}_0 + \check{\sigma}_1\check{\epsilon}_1) = 0$ on $\mathbb{C}$. Then

$$\begin{gathered}
(\check{\sigma}_0)_z + \frac{1}{\eta^2}\check{\sigma}_1 - \frac{\beta_z i}{2}\check{\sigma}_0 - \check{\sigma}_1 = 0,\\
(\check{\sigma}_1)_z - \frac{\beta_z^2}{4}\check{\sigma}_0 + \frac{\beta_z i}{2}\check{\sigma}_1 = 0,
\end{gathered}$$

Let $\check{\sigma}_1(z,\eta) = R(z,\eta)e^{-\beta_z iz/2}$, where $R = R(z,\eta)$ is a complex-valued function. The second equation becomes

$$-\frac{\beta_z^2}{4}\check{\sigma}_0 + R_z e^{-\beta_z iz/2} = 0.$$

Hence $\check{\sigma}_0 = 4R_z e^{-\beta_z iz/2}/\beta_z^2$. The first equation becomes

$$\frac{4e^{-\beta_z iz/2}}{\beta_z^2}\left[R_{zz} - \beta_z i R_z + \frac{\beta_z^2}{4}\left(\frac{1}{\eta^2} - 1\right)R\right] = 0.$$

Then

$$R(z,\eta) = \check{C}_0(z,\eta)e^{[\beta_z(1/\eta+1)i/2]z} + \check{C}_1(z,\eta)e^{[-\beta_z(1/\eta-1)i/2]z},$$

where $\check{C}_0(z,\eta)$ and $\check{C}_1(z,\eta)\colon \mathbb{C}\times(\mathbb{C}\setminus\{0\}) \to \mathbb{C}^2$ are anti-holomorphic maps with respect to z. Then

$$\begin{pmatrix}\check{\sigma}_0 & \check{\sigma}_1\end{pmatrix} = C_0(z,\eta)e^{[\beta_z i/(2\eta)]z} + C_1(z,\eta)e^{[-\beta_z i/(2\eta)]z}. \tag{5}$$

Let

$$\hat{\epsilon} = \begin{pmatrix}\hat{\epsilon}_0\\ \hat{\epsilon}_1\end{pmatrix} = \begin{pmatrix}\beta_{\bar{z}}i/4 & \beta_{\bar{z}}/4\\ 1 & 0\end{pmatrix}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix}.$$

Then

$$\tilde{\nabla}^{(0,1)}_{\eta^2}\hat{\epsilon} = \left\{\begin{pmatrix}\bar{\partial} & 0\\ 0 & \bar{\partial}\end{pmatrix} + \left[\eta^2\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \begin{pmatrix}-\beta_{\bar{z}}i/2 & -\beta_{\bar{z}}^2/4\\ -1 & \beta_{\bar{z}}i/2\end{pmatrix}\right](d\bar{z})\right\}\begin{pmatrix}\hat{\epsilon}_0\\ \hat{\epsilon}_1\end{pmatrix}.$$

In a similar fashion, we see that $\hat{\sigma}_0\hat{\epsilon}_0+\hat{\sigma}_1\hat{\epsilon}_1 = \hat{\sigma}_0(z,\eta)\hat{\epsilon}_0+\hat{\sigma}_1(z,\eta)\hat{\epsilon}_1$ is a solution to the differential equation $u^*\tilde{\nabla}^{(0,1)}_{\eta^2}(\hat{\sigma}_0\hat{\epsilon}_0+\hat{\sigma}_1\hat{\epsilon}_1) = 0$ if and only if

$$\begin{pmatrix}\hat{\sigma}_0 & \hat{\sigma}_1\end{pmatrix} = D_0(z,\eta)e^{(\beta_{\bar{z}}\eta i/2)\bar{z}} + D_1(z,\eta)e^{(-\beta_{\bar{z}}\eta i/2)\bar{z}}. \tag{6}$$

where $D_0(z,\eta)$ and $D_1(z,\eta)\colon \mathbb{C}\times(\mathbb{C}\setminus\{0\}) \to \mathbb{C}^2$ are holomorphic maps with respect to z.

We have

$$\begin{aligned}
\begin{pmatrix}\check{\sigma}_0 & \check{\sigma}_1\end{pmatrix}\begin{pmatrix}\check{\epsilon}_0\\ \check{\epsilon}_1\end{pmatrix} &= \begin{pmatrix}\check{\sigma}_0 & \check{\sigma}_1\end{pmatrix}\begin{pmatrix}\beta_z i/4 & -\beta_z/4\\ 1 & 0\end{pmatrix}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix}\\
&= \begin{pmatrix}\check{\sigma}_0 & \check{\sigma}_1\end{pmatrix}\begin{pmatrix}\beta_z i/4 & -\beta_z/4\\ 1 & 0\end{pmatrix}\begin{pmatrix}\beta_{\bar{z}}i/4 & \beta_{\bar{z}}/4\\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix}\hat{\epsilon}_0\\ \hat{\epsilon}_1\end{pmatrix} = \begin{pmatrix}\hat{\sigma}_0 & \hat{\sigma}_1\end{pmatrix}\begin{pmatrix}\hat{\epsilon}_0\\ \hat{\epsilon}_1\end{pmatrix}
\end{aligned}$$

Hence if $\psi_0+\psi_1 j$ is a parallel section of $u^*\tilde{\nabla}_{\eta^2}$, then

$$\psi_0+\psi_1 j = \sum_{m,n=0}^{1}\begin{pmatrix}p_{m,n,0} & p_{m,n,1}\end{pmatrix}e^{[(-1)^m\beta_z i/(2\eta)]z+[(-1)^n\beta_{\bar{z}}i\eta/2]\bar{z}}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix},$$

where $p_{m,n,l} = p_{m,n,l}(\eta)$ is a complex-valued map on $\mathbb{C}\setminus\{0\}$ $(l,m,n = 0,1)$. $\square$

8 Proof of Theorem 1.1

By Lemma 4.1, every Hamiltonian stationary Lagrangian torus in $\mathbb{C}^2$ whose image is contained in the hypersphere centered at the origin is a parallel section of a connection ∇_{η^2} in the associated family $\mathcal{C} = \left\{\nabla_{\eta^2} : |\eta| = 1\right\}$ of the trivial connection. Hence it is considered as a parallel section of $\tilde{\nabla}_{\eta^2}$ in

$\left\{\tilde{\nabla}_{\eta^2} : \nabla_{\eta^2} \in \mathcal{C}\right\}$. Let ψ_0^η and ψ_1^η be complex-valued functions on $\mathbb{C}$ such that $\psi^\eta = \psi_0^\eta \epsilon_0 + \psi_1^\eta \epsilon_1 \colon \mathbb{C} \to \mathbb{H}$ is parallel with respect to $\tilde{\nabla}_{\eta^2}$ and periodic with respect to $\tilde{\Lambda}_\delta$ in $\mathbb{C}$. By Lemma 7.1 and $|\eta| = 1$, we can assume that

$$\begin{aligned}
&\begin{pmatrix}\psi_0^\eta & \psi_1^\eta\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix}\\
&= \sum_{m,n=0}^{1} \begin{pmatrix}p_{m,n,0} & p_{m,n,1}\end{pmatrix} e^{[(-1)^m \beta_z \bar{\eta} i/2]z + [(-1)^n \beta_{\bar{z}} i\eta/2]\bar{z}} \begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix},
\end{aligned}$$

where $p_{m,n,l} = p_{m,n,l}(\eta)$ is a complex-valued function on $\mathbb{C}\setminus\{0\}$ $(l, m, n = 0, 1)$. We have

$$\begin{aligned}
\begin{pmatrix}\theta\\ j\theta\end{pmatrix} &= \begin{pmatrix}e^{\beta i/2} & ie^{-\beta i/2}\\ ie^{\beta i/2} & e^{-\beta i/2}\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix}\\
&= \begin{pmatrix}e^{\beta i/2} & ie^{-\beta i/2}\\ ie^{\beta i/2} & e^{-\beta i/2}\end{pmatrix}\begin{pmatrix}e^{-\beta i/2} & 0\\ 0 & e^{\beta i/2}\end{pmatrix}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix} = \begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}\tilde{\epsilon}_0\\ \tilde{\epsilon}_1\end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\begin{pmatrix}\psi_0^\eta & \psi_1^\eta\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix}\\
&= \frac{1}{2}\sum_{m,n=0}^{1} \begin{pmatrix}p_{m,n,0} & p_{m,n,1}\end{pmatrix}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix} e^{[(-1)^m \beta_z \bar{\eta} i/2]z + [(-1)^n \beta_{\bar{z}} i\eta/2]\bar{z}} \begin{pmatrix}\theta\\ j\theta\end{pmatrix}.
\end{aligned}$$

Then the equation $\nabla_{\eta^2}\psi^\eta = 0$ is equivalent to a system of equations

$$\begin{pmatrix}\psi_0^\eta & \psi_1^\eta\end{pmatrix}\begin{pmatrix}\epsilon_0\\ \epsilon_1\end{pmatrix} = \begin{pmatrix}\lambda_0 & \lambda_1\end{pmatrix}\begin{pmatrix}\theta\\ j\theta\end{pmatrix},$$

$$\mathcal{D}\begin{pmatrix}\lambda_0\\ \lambda_1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \tag{7}$$

$$\begin{aligned}
\begin{pmatrix}\lambda_0\\ \lambda_1\end{pmatrix} &= \begin{pmatrix}F_{00}(\eta)\\ F_{01}(\eta)\end{pmatrix} e^{i\,\mathrm{Re}[(\beta_z\bar{\eta})z]} + \begin{pmatrix}F_{10}(\eta)\\ F_{11}(\eta)\end{pmatrix} e^{i\,\mathrm{Im}[(\beta_z\bar{\eta})z]}\\
&+ \begin{pmatrix}F_{20}(\eta)\\ F_{21}(\eta)\end{pmatrix} e^{-i\,\mathrm{Im}[(\beta_z\bar{\eta})z]} + \begin{pmatrix}F_{30}(\eta)\\ F_{31}(\eta)\end{pmatrix} e^{-i\,\mathrm{Re}[(\beta_z\bar{\eta})z]},
\end{aligned} \tag{8}$$

where $F_{nm} = F_{nm}(\eta)$ is a map from $\mathbb{C}\setminus\{0\}$ to $\mathbb{C}$ $(n = 0, 1, 2, 3,\ m = 0, 1)$. From the equation (7), we have

$$\begin{aligned}
F_{00} + \bar{\eta}iF_{01} = 0,&\ \ -\eta iF_{00} + F_{01} = 0,\\
F_{10} + \bar{\eta}iF_{11} = 0,&\ \ \eta iF_{10} + F_{11} = 0,\\
F_{20} - \bar{\eta}iF_{21} = 0,&\ \ -\eta iF_{20} + F_{21} = 0,\\
F_{30} - \bar{\eta}iF_{31} = 0,&\ \ \eta iF_{30} + F_{31} = 0.
\end{aligned}$$

The equation (8) becomes

$$F_{01} = \eta i F_{00}, \;\; F_{31} = -\eta i F_{30}, \;\; F_{10} = F_{11} = F_{20} = F_{21} = 0.$$

We have

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \eta i \end{pmatrix} F_{00}(\eta) e^{i\,\mathrm{Re}[(\beta_z \bar{\eta})z]} + \begin{pmatrix} 1 \\ -\eta i \end{pmatrix} F_{30}(\eta) e^{-i\,\mathrm{Re}[(\beta_z \bar{\eta})z]}.$$

By the periodicity with respect to $\tilde{\Lambda}_\delta$, we have

$$\mathrm{Re}(\beta_z \bar{\eta}) = m\pi, \;\; \mathrm{Re}(\beta_z \bar{\eta}\delta) = n\pi \;\; (n, m \in \mathbb{Z}).$$

We see that $m^2 + n^2 \neq 0$. Indeed, if $m = n = 0$, then $\delta \in \mathbb{R}$. This contradicts the assumption $\delta_1 > 0$. Then

$$\begin{aligned} \eta &= \frac{\pi(m\delta - n)}{\beta_{\bar{z}}\delta_1 i} = -\frac{\beta_z \delta_1 i}{\pi(m\bar{\delta} - n)} \\ &= \frac{m\delta_1 - (m\delta_0 - n)i}{r\delta_1 - (r\delta_0 - s)i} = \frac{r\delta_1 + (r\delta_0 - s)i}{m\delta_1 + (m\delta_0 - n)i}, \\ \mathrm{Re}(\beta_z \bar{\eta} z) &= \mathrm{Re}\left\{ \frac{\pi[m\delta_1 + (m\delta_0 - n)i](x + yi)}{\delta_1} \right\} = \pi\left(mx - \frac{m\delta_0 - n}{\delta_1} y \right). \end{aligned}$$

If $(m, n) = (\pm r, \pm s)$, then this equation always holds. Since $\eta = \pm 1$, we have $\nabla_{\eta^2} = \nabla_1 = d$. Then the connection ∇_{η^2} is not a connection of a Hamiltonian stationary Lagrangian torus.

We assume that $(m, n) \neq (\pm r, \pm s)$. Then $\pi^2 |m\delta - n|^2 = |\beta_z| \delta_1^2$. This equation is equivalent to the equation

$$m^2\left(\delta_0^2 + \delta_1^2\right) - 2mn\delta_0 + n^2 = r^2\delta_1^2 + (r\delta_0 - s)^2.$$

Hence

$$\left(m^2 - r^2\right)\delta_0^2 - 2(mn - rs)\delta_0 + (m^2 - r^2)\delta_1^2 + n^2 - s^2 = 0.$$

If δ_0, δ_1, r, s, m, and n satisfy the above equation, then

$$
\begin{aligned}
&\psi_0^\eta + \psi_1^\eta j \\
=&\Big[F_{00}(\eta)e^{i\,\mathrm{Re}[(\beta_z\bar\eta)z]}(1+\eta ij) \\
&+F_{30}(\eta)e^{-i\,\mathrm{Re}[(\beta_z\bar\eta)z]}(1-\eta ij)\Big]\left(e^{\beta i/2}+ie^{-\beta i/2}j\right) \\
=&F_{00}(\eta)e^{i\,\mathrm{Re}[(\beta_z\bar\eta)z]}\left[(1-\eta)e^{\beta i/2}+(1+\eta)ie^{-\beta i/2}j\right] \\
&+F_{30}(\eta)e^{-i\,\mathrm{Re}[(\beta_z\bar\eta)z]}\left[(1+\eta)e^{\beta i/2}+(1-\eta)ie^{-\beta i/2}j\right] \\
=&F_{00}(\eta)\left[(1-\eta)e^{i\,\mathrm{Re}[(\beta_z\bar\eta)z+\beta/2]}+(1+\eta)ie^{i\,\mathrm{Re}[(\beta_z\bar\eta)z-\beta/2]}j\right] \\
&+F_{30}(\eta)\left[(1+\eta)e^{-i\,\mathrm{Re}[(\beta_z\bar\eta)z-\beta/2]}+(1-\eta)ie^{-i\,\mathrm{Re}[(\beta_z\bar\eta)z+\beta/2]}j\right] \\
=&F_{00}(\eta)\left[(1-\eta)e^{i\,\mathrm{Re}[(\beta_z\bar\eta)z+\beta/2]}+(1+\eta)ie^{i\,\mathrm{Re}[(\beta_z\bar\eta)z-\beta/2]}j\right] \\
&+F_{30}(\eta)ij\left[(1-\bar\eta)e^{i\,\mathrm{Re}[(\beta_z\bar\eta)z+\beta/2]}-(1+\bar\eta)ie^{i\,\mathrm{Re}[(\beta_z\bar\eta)z-\beta/2]}j\right] \\
=&\left(F_{00}(\eta)-\eta F_{30}(\eta)ij\right)\left[(1-\eta)e^{i\,\mathrm{Re}[(\beta_z\bar\eta)z+\beta/2]}+(1+\eta)ie^{i\,\mathrm{Re}[(\beta_z\bar\eta)z-\beta/2]}j\right].
\end{aligned}
$$

Hence ψ^η is a homogeneous torus.

Remark 1. We give an explicit formula for ψ^η. Since

$$
\begin{aligned}
\mathrm{Re}(\beta_z\bar\eta z)+\frac{\beta}{2} &= \pi\left[(m+r)x-\frac{(m+r)\delta_0-(n+s)}{\delta_1}y\right], \\
\mathrm{Re}(\beta_z\bar\eta z)-\frac{\beta}{2} &= \pi\left[(m-r)x-\frac{(m-r)\delta_0-(n-s)}{\delta_1}y\right], \\
1-\eta &= \frac{(r-m)\delta_1-[(r-m)\delta_0-(n-s)]i}{r\delta_1-(r\delta_0-s)i}, \\
1+\eta &= \frac{(r+m)\delta_1-[(r+m)\delta_0-(n+s)]i}{r\delta_1-(r\delta_0-s)i},
\end{aligned}
$$

we have

$$
\begin{aligned}
\psi^\eta =& \left(F_{00}(\eta)-\frac{m\delta_1-(n\delta_0-s)}{r\delta_1-(r\delta_0-s)}F_{30}(\eta)ij\right) \\
&\times\frac{1}{r\delta_1-(r\delta_0-s)i} \\
&\times\Big[\{(r-m)\delta_1-[(r-m)\delta_0-(s-n)]i\}\,e^{\pi\{(m+r)\delta_1x-[(m+r)\delta_0-(n+s)]y\}i/\delta_1} \\
&+\{(r+m)\delta_1-[(r+m)\delta_0-(s+n)]i\}\,ie^{\pi\{(m-r)\delta_1x-[(m-r)\delta_0-(n-s)]y\}i/\delta_1}j\Big].
\end{aligned}
$$

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